MA 681, Spring 2016

Assignment 12.

Isolated singular points

This assignment is due Wednesday, April 22. Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

(1) Find and classify singular points (i.e. in each case decide whether the point is removable, a pole of order N, essential, or not an isolated singular point), including infinity, of the following functions:
(a) $\frac{1}{z-z^3}$, (b) $\frac{1}{(z^2+4)^2}$, (c) $\frac{e^z}{1+z^2}$, (d) $\frac{z^2+1}{e^z}$, (e) $\frac{1}{e^z-1}-\frac{1}{z}$,
(f) e^{-1/z^2}, (g) $\cot\frac{1}{z}$, (h) $e^{-z}\cos\frac{1}{z}$, (i) $e^{\cot\frac{1}{z}}$, (j) $\cot\frac{1}{z}-\frac{1}{z}$,
(k) $\sin\left(\frac{1}{\cos\frac{1}{z}}\right)$.
(*Hint:* Among other things, the problems below may help.)

(2) Suppose $z_0 \in \mathbb{C}$ is an isolated singular point of the function f of a given type (removable, pole of order N, essential). Show that z_0 is an isolated singular point of
(a) $g(z) = 1/f(z)$ (here additionally assume that $f(z)$ has no zeros in some punctured neighborhood of z_0),
(b) $h(z) = f^2(z)$
and find its type.

(3) (a) Suppose $f(z)$ and $g(z)$ have poles or order m and n, respectively, at a point $z_0 \in \mathbb{C}$, with $m \neq n$. Show that z_0 is an isolated singular point of $f+g$ and find its type.
(b) Same question when $m = n$.

(4) (a) Suppose $f(z)$ is analytic and nonzero at $z_0 \in \mathbb{C}$, and that $g(z)$ has a non-removable isolated singularity of a given type at z_0. Show that z_0 is an isolated singular point of fg and find its type.
(b) Same question when z_0 is a pole of order N of f.
(c) Can anything be asserted about the type of z_0 for fg if f and g have essential singularity at z_0?